COMPRESSION OF AN ELASTIC SPHERE WITH A NONCONCENTRIC SPHERICAL CAVITY

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The above problem is solved with the aid of solutions of the static equation of the theory of elasticity [1]

$$\Delta \mathbf{u} + \frac{1}{1-2\sigma} \operatorname{grad} \operatorname{div} \mathbf{u} = 0 \qquad \begin{pmatrix} \mathbf{u} \text{ is the displacement vector} \\ \sigma \text{ is the Poisson's ratio} \end{pmatrix} \qquad (0.1)$$

In the process of solution it became necessary to find new formulas for the transformation of solutions of Equation (0.1) from one center to another (transfer formulas), which are not given in [1]. The solution of the boundary value problem is sought in the form of series with unknown coefficients which are determined, as in [1], from an infinite system of linear algebraic equations [2].

1. We are going to make use of the following axisymmetrical solutions taken from [1]: the so called exterior solutions $\mathbf{u}_{l0}$ and $\mathbf{v}_{l0}$, and the interior ones $\mathbf{p}_{l0}$ and $\mathbf{q}_{l0}$. They have the form (r and θ are spherical coordinates, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the corresponding unit vectors)

$$\begin{aligned}
\mathbf{u}_{l0}(r, \theta) &= r^{-l}\left[\beta_l P_l(\cos\theta)\,\mathbf{e}_r + \delta_l \frac{\partial}{\partial\theta} P_l(\cos\theta)\,\mathbf{e}_\theta\right] \qquad (l = 1, 2, \ldots) \\
\mathbf{v}_{l0}(r, 0) &= r^{-(l+2)}\left[P_l(\cos\theta)\,\mathbf{e}_r - \frac{1}{l+1}\frac{\partial}{\partial\theta} P_l(\cos\theta)\,\mathbf{e}_\theta\right] \qquad (l = 0, 1, 2, \ldots) \\
\mathbf{p}_{l0}(r, \theta) &= r^{l+1}\left[d_l P_l(\cos\theta)\,\mathbf{e}_r - \gamma_l \frac{\partial}{\partial\theta} P_l(\cos\theta)\,\mathbf{e}_\theta\right] \qquad (l = 0, 1, 2, \ldots) \\
\mathbf{q}_{l0}(r, \theta) &= r^{l-1}\left[P_l(\cos\theta)\,\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta} P_l(\cos\theta)\,\mathbf{e}_\theta\right] \qquad (l = 1, 2, \ldots)
\end{aligned} \qquad (1.1)$$

Here P_l are the normalized Legendre polynomials

$$\begin{aligned}
\alpha_l &= \frac{2\rho - (1-\rho)\,l}{2l+3}, \qquad \beta_l = \frac{(\rho-1)\,l - (\rho+1)}{2l-1} \qquad \left(\rho = \frac{1-2\sigma}{2(1-\sigma)}\right) \\
\gamma_l &= \frac{(1-\rho)\,l + 3 - \rho}{(2l+3)(l+1)}, \qquad \delta_l = \frac{l^2(1-\rho) - l\,(1+\rho) - 2}{(2l-1)\,l\,(l+1)}
\end{aligned} \qquad (1.2)$$

The transfer formulas (*) which express exterior solutions again in terms of exterior solutions, and transfer formulas for the interior solutions are found in a manner analogous to that shown in [1], but in this case the following transfer formulas for the spherical functions are employed (see [3]) (Fig.1)

$$\frac{P_l(\cos\theta_2)}{r_2^{l+1}} = \sum_{k=l}^{\infty} \frac{P_k(\cos\theta_1)}{r_1^{k+1}} d^{k-l} \frac{k!}{l!\,(k-l)!} \qquad (r_1 > d)$$

$$r^1 P_l(\cos\theta_1) = \sum_{k=0}^{l} r_2^k P_k(\cos\theta_2)\, d^{l-k} \frac{l!}{k\,(l-k)!}$$

As a result we obtain

$$\mathbf{u}_{l0}(r_2, \theta_2) = \sum_{k=l}^{\infty} d^{k-l} \pi_{lk} \mathbf{u}_{k0}(r_1, \theta_1) + \sum_{k=l-1}^{\infty} d^{k-l+2} \rho_{lk} \mathbf{v}_{k0}(r_1, \theta_1) \qquad (r_1 > d)$$

$$\mathbf{v}_{l0}(r_2, \theta_2) = \sum_{k=l}^{\infty} d^{k-l} \nu_{lk} \mathbf{v}_{k0}(r_1, \theta_1) \qquad (r_1 > d) \tag{1.3}$$

$$\mathbf{q}_{l0}(r_1, \theta_1) = \sum_{k=1}^{l} \varkappa_{lk} d^{l-k} \mathbf{q}_{k0}(r_2, \theta_2)$$

$$\mathbf{p}_{l0}(r_1, \theta_1) = \sum_{k=0}^{l} \psi_{lk} d^{l-k} \mathbf{p}_{k0}(r_2, \theta_2) + \sum_{k=1}^{l+1} \omega_{lk} d^{l-k+2} \mathbf{q}_{k0}(r_2, \theta_2) \tag{1.4}$$

where

$$\pi_{lk} = \frac{k!}{l!\,(k-l)!}, \quad \varkappa_{lk} = \frac{(l+1)!}{(k-1)!\,(l-k)!}, \quad \nu_{lk} = \frac{(k+1)!}{(l+1)!\,(k-l)!}, \quad \psi_{lk} = \frac{l!}{k!\,(l-k)!}$$

$$\rho_{lk} = \frac{(k+1)!}{l!\,(k-l+1)!}\left[\frac{(1-\rho)(2kl+5l+k+3)}{(2l+1)(2k-5)} + \frac{2l(\rho l+l+1)}{(4l^2-1)(k-l+2)} + \frac{2(\rho k+2\rho+k+3)(k+2)}{(k-l+2)(4k^2+16k+15)}\right]$$

$$\omega_{lk} = \frac{l!}{(k-1)!\,(l-k+1)!}\left[\frac{(\rho-1)(2lk-3l+k-1)}{(2l+1)(2k+3)} + \frac{2(l+1)(\rho l+l+\rho)}{(2l+3)(2l+1)(l-k+2)} - \frac{2(\rho k-k-\rho-2)(k-1)}{(2k-1)(2k-3)(l-k+2)}\right]$$

2. Consider a boundary value problem of the static theory of elasticity for a sphere with a nonconcentric spherical cavity (Fig.1), with the following boundary conditions in spherical coordinates.

On the sphere 1 (uniform compression of the sphere by a given normal force)

$$\sigma_{rr} = \text{const} = p, \qquad \sigma_{r\theta} = \sigma_{r\varphi} = 0 \tag{2.1}$$

on the sphere 2 (free surface)

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\varphi} = 0 \tag{2.2}$$

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*) These formulas were derived by Ts.A. Kapelian.

Bearing in mind the axial symmetry of the problem we will seek the solution (the displacement vector) in the form

$$\mathbf{u} = \sum_{l=0}^{\infty} A_l \frac{p_{l0}(r_1, \theta_1)}{R_1^{\,l}} + \sum_{l=1}^{\infty} B_l \frac{q_{l0}(r_1, \theta_1)}{R_1^{\,l-2}} +$$

$$+ \sum_{l=1}^{\infty} c_l R_2^{\,l+1} \mathbf{u}_{l0}(r_2, \theta_2) + \sum_{l=0}^{\infty} D_l R_2^{\,l+3} \mathbf{v}_{l0}(r_2, \theta_2) \qquad (2.3)$$

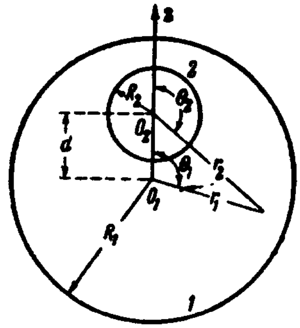

Fig. 1

N o t e . The coefficients of $\mathbf{u}_{00}$ and $\mathbf{q}_{00}$ are set equal to zero, since the denominators of those solutions vanish.

We shall require that the boundary conditions be satisfied (remembering that in the axisymmetrical solutions $\sigma_{r\varphi} \equiv 0$).

In order to satisfy the boundary conditions on sphere 1, we transfer $\mathbf{u}$ completely to the variables r_1 and θ_1 . Using the transfer formulas (1.3) and changing the order of summation we obtain

$$\mathbf{u}(r_1, \theta_1) = \sum_{k=0}^{k} A_k \frac{\mathbf{p}_{k0}(r_1, \theta_1)}{R_1^{\,k}} + \sum_{k=1}^{\infty} B_k \frac{\mathbf{q}_{k0}(r_1, \theta_1)}{R_1^{\,k-2}} +$$

$$+ \sum_{k=1}^{\infty} \mathbf{u}_{k0}(r_1, \theta_1) \sum_{l=1}^{k} d^{k-l} \pi_{lk} R_2^{\,l+1} C_l + \sum_{k=0}^{\infty} \mathbf{v}_{k0}(r_1, \theta_1) \sum_{l=0}^{k} d^{k-l} \nu_{lk} R_2^{\,l+3} D_l \qquad (r_1 > d) \quad (2.4)$$

Now, let us require that

$$\sigma_{rr} = p, \qquad \sigma_{r\theta} = 0 \qquad \text{for } r_1 = R_1 \qquad (2.5)$$

where σ_{rr} and $\sigma_{r\theta}$ are computed by the well-known formulas (E is the modulus of elasticity)

$$\sigma_{rr} = \frac{E}{1+\sigma}\left(\frac{\partial u_r}{\partial r} + \frac{\sigma}{1-2\sigma}\operatorname{div}\mathbf{u}\right), \quad \sigma_{r\theta} = \frac{1}{2}\,\frac{E}{1+\sigma}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \quad (2.6)$$

Noting that [1]

$$\operatorname{div}\mathbf{u}_{l0} = \frac{P_l}{r^{l+1}}, \qquad \operatorname{div}\mathbf{v}_{l0} = 0, \qquad \operatorname{div}\mathbf{p}_{l0} = P_l r^l, \qquad \operatorname{div}\mathbf{q}_{l0} = 0 \qquad (2.7)$$

and making use of (1.1), (2.6) and (2.7), from the boundary conditions (2.5) we obtain

$$\frac{1}{k+2}\left[(k+1)\alpha k + \frac{\sigma}{1-2\sigma}\right]A_k + \frac{k-1}{k-2}B_k + \sum_{l=1}^{k}\frac{1}{k+2}\left(\frac{\sigma}{1-2\sigma} - k\beta_k\right)\times$$

$$\times\frac{d^{k-l}R_2^{\,l+1}}{R_1^{\,k+1}}\pi_{lk}C_l - \sum_{l=0}^{k}\frac{d^{k-l}R_2^{\,l+1}}{R_1^{\,l+3}}\nu_{lk}D_l = b_k$$

$$(2.8)$$

$$\frac{k^2(\rho-1) + 2k(\rho-1) + \rho}{(2k+3)(k+1)}A_k + \frac{k-1}{k}B_k + \sum_{l=1}^{k}\frac{(\rho-1)k^2+1}{k(2k-1)}\,\frac{d^{k-l}R_2^{\,l+1}}{R_1^{\,k+1}}\,\pi_{lk}C_l +$$

$$+ \sum_{l=0}^{k}\frac{1}{2}\,\frac{k+2}{k+1}\,\frac{d^{k-l}R_2^{\,l+3}}{R_1^{\,k+3}}\,\nu_{lk}D_l = 0 \qquad (k = 0, 1, 2, \ldots)$$

where

$$b_k = \frac{(1+\sigma)\,p}{E\,(k+2)}\,\delta_{k0} \qquad (\delta_{k0} = 0, \quad \text{if } k \neq 0, \ \delta_{00} = 1) \tag{2.9}$$

By an analogous procedure (but using the transfer formulas (1.4) in this case), from the boundary conditions on sphere 2

$$\sigma_{rr} = \sigma_{r\theta} = 0 \quad \text{for } r_2 = R_2 \tag{2.10}$$

we obtain

$$\frac{1}{k+2}\left(\frac{\sigma}{1-2\sigma} - k\beta_k\right) C_k - D_k + \sum_{l=k}^{\infty} \frac{1}{k+2}\left[(k+1)\,\alpha_k + \frac{\sigma}{1-1\sigma}\right] \times$$

$$\times \frac{d^{l-k}}{R_1{}^l R_2{}^{-k}}\,\psi_{lk} A_l + \sum_{l=k-1}^{\infty} \frac{(k-1)}{(k+2)} \frac{d^{l-k+2}}{R_1{}^l R_2{}^{-(k-2)}}\,\omega_{lk} A_l + \sum_{l=k}^{\infty} \frac{(k-1)\,d^{l-k}}{(k+2)\,R_1{}^{l-2} R_2{}^{-(k-2)}}\,\varkappa_{lk} B_l = 0$$

$$\tag{2.11}$$

$$\frac{[(\rho-1)\,k^2+1]}{k\,(2k-1)}\,C_k + \frac{1}{2}\,\frac{k+2}{k+1}\,D_k + \sum_{l=k}^{\infty} \frac{k^2(\rho-1) + 2k\,(\rho-1) + \rho}{(2k+3)\,(k+1)}\;\frac{d^{l-k}}{R_1{}^l R_2{}^{-k}}\,A_l \psi_{lk} +$$

$$+ \sum_{l=k-1}^{\infty} \frac{k-1}{k}\,\frac{d^{l-k+2}}{R_1{}^l R_2{}^{-(k-2)}}\,\omega_{lk} A_l + \sum_{l=k}^{\infty} \frac{(k-1)\,d^{l-k}}{k R_1{}^{l-2} R_2{}^{-(k-2)}}\,\varkappa_{lk} B_l = 0 \qquad (k = 0, 1, 2, \ldots)$$

Formulas (2.8) and (2.11) furnish an infinite system of linear algebraic equations for the determination of the unknown coefficients A_l, B_l, C_l and D_l.

N o t e 1 . To obtain equations which correspond to $k = 0$, one should set $k = 0$ in the system, and discard the expressions which become meaningless for $k = 0$, also bearing in mind that we have set: $B_0 = C_0 = 0$ from the very beginning (see Note after (2.3)).

N o t e 2 . It is easy to see that the system (2.8),(2.11) does not contain the coefficient B_1, since the factors at B_1 ($k = 1$) become zero. Choosing B_1 arbitrarily, from Formula (2.3) we obtain solutions which differ by a constant vector, since B_1 is the coefficient of $\mathbf{q}_{10}$ = const (it follows from (1.1) that $\mathbf{q}_{10} = \mathbf{e}_z$). This is consistent with the fact that the solution of our boundary value problem is unique up to an additive constant.

Designating the linear combinations of coefficients A_l, B_l, C_l and D_l, appearing outside of the summation signs, by

$$z_{4k},\ z_{4k+1},\ z_{4k+2},\ z_{4k+3}\ (k = 0, 1, 2, \ldots),$$

we obtain a system in canonical form

$$z_k + \sum_{l=0}^{\infty} c_{kl} z_l = b_k \qquad (k = 0, 1, 2, \ldots) \tag{2.12}$$

Let us show that

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{kl}^2 < \infty \tag{2.13}$$

First we note that (see [4])

$$\sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{k!}{l!\,(k-l)!}\,u_1{}^l u_2{}^{k-l} = \sum_{k=0}^{\infty} (u_1 + u_2)^k < \infty \tag{2.14}$$

$$\sum_{k=0}^{\infty}\sum_{l=k}^{\infty}\frac{l!}{k!\,(l-k)!}u_1^k u_2^{l-k}=\sum_{k=0}^{\infty}\sum_{l_1=0}^{\infty}\frac{(l_1+k)!}{l_1!\,k!}u_1^k u_2^{l_1}=\sum_{k=0}^{\infty}\frac{u_1^k}{(1-u_2)^{k+1}}<\infty \qquad \begin{matrix}(2.14)\\ \text{cont.}\end{matrix}$$

provided that

$$u_1>0, \qquad u_2>0, \qquad u_1+u_2<1 \tag{2.15}$$

If we make a designation

$$\frac{d}{R_1}=u_1, \qquad \frac{R_2}{R_1}=u_2$$

the inequalities (2.15) will be satisfied and the coefficients in our system will differ from the expressions under the double summation signs in the equalities (2.14) by bounded factors only. Therefore, the following will be true

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}|c_{kl}|<\infty$$

Hence, (2.13) is also satisfied. Thus the matrix of coefficients c_{kl} of the system under consideration is a completely continuous operator in the Hilbert space l_2. It follows from (2.9) that the absolute term b_k also belongs to that space. Hence, the Fredholm's alternative is valid for the system (2.8),(2.11). However, in view of Note 2, the corresponding homogeneous system, which is obtained for $p = 0$, i.e. for zero boundary conditions (see (2.1) and (2.2)), can only have a trivial solution.

Hence it follows that the system (2.8),(2.11) has the unique bounded solution for arbitrary right-hand sides. This solution can be found by the method of truncation or reduction, and also by the method of successive approximations (see [4], where the last statement is proved for an analogous system).

The proof that the series in (2.3) converge and give the solution of the boundary value problem under consideration is carried out in the manner similar to that of [1].

The boundary value problems for the same domain with more general boundary conditions can be solved in an analogous way (see (2.2) in [1]).

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